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Relativistic effects in the production of pseudoscalar and vector doubly heavy mesons from e^+e^- annihilation

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On the basis of the perturbative QCD and the relativistic quark model we investigate the relativistic and bound state effects in the production processes of a pair of S-wave doubly heavy mesons with opposite charge conjugation consisting of b and c quarks. All possible relativistic corrections in the production amplitude including the terms connected with the transformation law of the bound state wave function to the reference frame of the moving pseudoscalar $\mathcal{P}-$ and vector $\mathcal{V}-$ mesons are taken into account. We obtain a growth of the cross section for the reaction $e^+ + e^- \to J/\Psi + \eta_c$ due to considered effects by a factor $2 \div 2.5$ in the range of the center-of-mass energy $\sqrt{s} = 6 \div 12$ GeV.

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I. INTRODUCTION

Different processes of the production and decay of heavy mesons consisting of heavy b and c quarks provide the means for revealing the role of the color and spin quark forces. The aim of many present experiments consists in the increase of experimental accuracy what is important for the detailed comparison of different existing theoretical approaches to the heavy quark problems [1, 2, 3, 4]. The exclusive production of a pair of doubly heavy mesons with c-quarks in e^+e^- annihilation has attracted considerable attention in the last years. This is due to the fact that the cross section of the process $e^+ + e^- \to J/\Psi + \eta_c$ which was measured in the experiments on Babar and Belle detectors at the energy $\sqrt{s} = 10.6$ GeV

$$\sigma(e^+e^- \to J/\Psi + \eta_c) \times \mathcal{B}(\eta_c \to \geq 2\ charged) = \begin{cases} 25.6 \pm 2.8 \pm 3.4 & [5,6] \\ 17.6 \pm 2.8^{+1.5}_{-2.1} & [7] \end{cases} \tag{1}$$

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leads to a discrepancy with the theoretical calculation in the framework of nonrelativistic QCD (NRQCD) by an order of magnitude [5, 6, 7, 8, 9, 10]. This conclusion is based on calculations in which the relative momenta of heavy quarks and bound state effects in the production amplitude were not taken into account. A set of calculations was performed to improve the nonrelativistic approximation for the process. In particular, relativistic corrections to the cross section $\sigma(e^+e^- \to J/\Psi\eta_c)$ were considered in a color singlet model in Ref.[8] using the methods of NRQCD [11]. It was obtained here that the relativistic corrections increase the cross section by a factor 2.4 for the production $J/\Psi + \eta_c$. Another attempt to take into account the relativistic corrections was done in the framework of the light-cone formalism [12, 13, 14]. Here it was shown that the discrepancy between the experiment and the theory can be eliminated completely by considering the intrinsic motion of heavy quarks forming the doubly heavy mesons. Thereupon, perturbative corrections of order α_s to the production amplitude were calculated in Ref.[15] increasing the cross section by a factor 1.8. On account of different values of relativistic corrections obtained in Refs.[8, 12, 13, 14] and the importance of a relativistic consideration of the process $e^+ + e^- \to J/\Psi + \eta_c$ in solving the doubly heavy meson production problem, we have performed a new investigation of relativistic and bound state effects. As in our recent papers [16, 17], this investigation is based on the relativistic quark model which provides the solution in many tasks of heavy quark physics. In the above quoted papers Refs.[16, 17] we have demonstrated how the original amplitude, describing the physical process, must be transformed in order to preserve the relativistic plus bound state corrections connected with the one-particle wave functions and the wave function of a two-particle bound state. In the present paper we shall extend the method of Refs.[16, 17] to the case of the production of a pair $(\mathcal{P} + \mathcal{V})$ of doubly heavy mesons containing quarks of different flavours b and c. In particular, we will consider the internal motion of heavy quarks in both produced pseudoscalar $\mathcal{P}$ and vector $\mathcal{V}$ mesons. The paper is organized as follows: In Sec.II we present the general formalism and the basic relations of our method which are required in order to formulate the relativistic amplitude for the production of doubly heavy mesons. In Sec.III we derive the analytical expressions for the corresponding cross sections and make numerical estimations exploiting the relativistic quark model. Conclusions and discussion of the results are given in Sec.IV. The construction of the quasipotential heavy quark distribution amplitude in the light-front variables is included in the Appendix.

II. GENERAL FORMALISM

The production of a pair of doubly heavy mesons in a color singlet model from e^+e^- annihilation contains two stages after the transition of the virtual photon γ^* into a quark-antiquark pair $(Q_1\bar{Q}_1)$. In the first stage examined on the basis of perturbative quantum chromodynamics (QCD), one of the heavy quarks (Q_1 or $\bar{Q}_1$) emits a gluon with sufficiently large energy $\sim \sqrt{s}$ which then transforms to another heavy quark-antiquark pair $(Q_2\bar{Q}_2)$. These four quarks can combine with a definite probability into a pair of S - wave pseudoscalar $(\bar{Q}_1Q_2)_{S=0}$ and vector $(Q_1\bar{Q}_2)_{S=1}$ doubly heavy mesons. The second nonperturbative stage of this process involves the formation of heavy quark bound states from heavy quarks. In the quasipotential approach to the relativistic quark model we can express the invariant transition amplitude for the described process as a simple convolution of a perturbative production amplitude of four heavy quarks $\mathcal{T}(p_1, p_2; q_1, q_2)$, projected onto the positive energy states, and the quasipotential wave functions of a vector meson $\Psi_{\mathcal{V}}(p, P)$ and a pseudoscalar meson $\Psi_{\mathcal{P}}(q,Q)$ [18]:

$$\mathcal{M}(p_-,p_+,P,Q)=\int\frac{d\mathbf{p}}{(2\pi)^3}\bar\Psi_{\mathcal{V}}(p,P)\int\frac{d\mathbf{q}}{(2\pi)^3}\bar\Psi_{\mathcal{P}}(q,Q)\cdot\mathcal{T}_\beta(p_1,p_2;q_1,q_2)\bar v(p_+)\gamma^\alpha u(p_-)\frac{4\pi\alpha g^{\alpha\beta}}{r^2+i\epsilon}, \tag{2}$$

where p_-, p_+ are four momenta of the electron and positron, $r^2=s=(p_-+p_+)^2$, p_1, p_2 are four-momenta of Q_1 and $\bar Q_2$ quarks forming the vector doubly heavy meson $\mathcal{V}$; q_1, q_2 are four momenta of $\bar Q_1$ and Q_2 forming the pseudoscalar doubly heavy meson $\mathcal{P}$. They are defined in terms of total momenta $P(Q)$ and relative momenta $p(q)$ as follows:

$$p_{1,2}=\eta_{1,2}P\pm p,\quad (p\cdot P)=0,\quad \eta_{1,2}=\frac{E^{\mathcal{V}}_{1,2}}{M_{\mathcal{V}}}=\frac{M^2_{\mathcal{V}}-m^2_{2,1}+m^2_{1,2}}{2M^2_{\mathcal{V}}}, \tag{3}$$

$$q_{1,2}=\rho_{1,2}Q\pm q,\quad (q\cdot Q)=0,\quad \rho_{1,2}=\frac{E^{\mathcal{P}}_{1,2}}{M_{\mathcal{P}}}=\frac{M^2_{\mathcal{P}}-m^2_{2,1}+m^2_{1,2}}{2M^2_{\mathcal{P}}}, \tag{4}$$

where $M_{\mathcal{V}}=m_1+m_2+W_{\mathcal{V}}$, $M_{\mathcal{P}}=m_1+m_2+W_{\mathcal{P}}$ are the masses of vector and pseudoscalar mesons consisting of heavy quarks.

Different color-spin nonperturbative factors entering the amplitude $\mathcal{M}(p_-,p_+,P,Q)$ control the production of doubly heavy mesons. In this process the gluon virtuality is large $k^2\gg\Lambda^2_{QCD}$, and the strong interaction constant is small $\alpha_s\ll 1$. There exist four production amplitudes in the leading order over α_s as explained below in Fig.1. In a color singlet model the first amplitude $\mathcal{T}_{1,\beta}(p_1,p_2;q_1,q_2)$ takes the form:

$$\mathcal{T}_{1,\beta}(p_1,p_2;q_1,q_2)=\frac{16\pi\alpha_s}{3}\bar u_1(p_1)\gamma_\mu\frac{(\hat r-\hat q_1+m_1)}{(r-q_1)^2-m_1^2+i\epsilon}\gamma_\beta v_1(q_1)\bar u_2(q_2)\gamma_\nu v_2(p_2)D_{\mu\nu}(k). \tag{5}$$

The color factor $\delta_{il}\delta_{kj}(T^a)_{ij}(T^a)_{kl}/3=4/3$ is already extracted in the amplitude (5). To calculate relativistic effects we have to keep all factors in the amplitude (2) with the relative motion momenta p and q. We have to take into account also the bound state corrections which are determined by the binding energies $W_{\mathcal{V}}$ and $W_{\mathcal{P}}$.

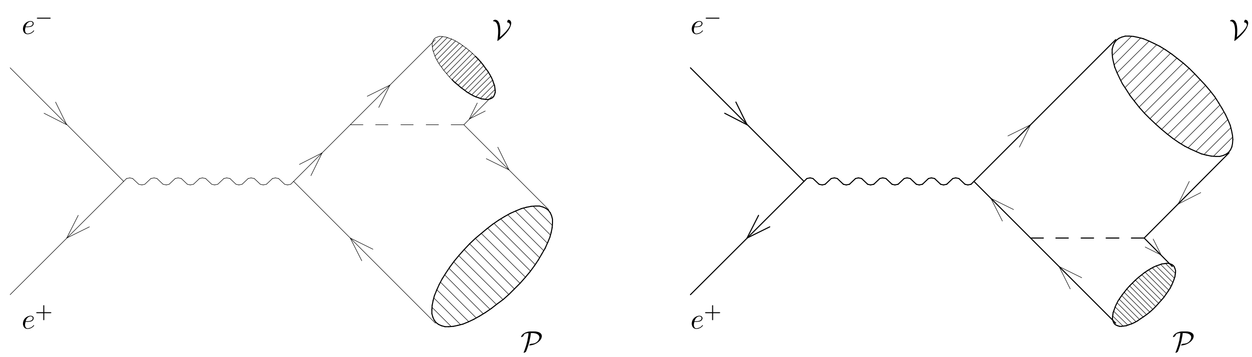

FIG. 1: The Feynman diagrams for the production of a pair of doubly heavy mesons $(\mathcal{P}+\mathcal{V})$ in e^+e^- annihilation. The wave line corresponds to the photon and the dashed line corresponds to the gluon. Two other diagrams can be obtained by permutations.

Clearly, in order to calculate the matrix element (2), one needs explicit expressions for bound state wave functions. We construct such wave functions by solving a relativistic quasipotential equation to the desired accuracy in the center-of-mass (CM) frame. But in the matrix element (2) the final meson states have different total momenta P and Q. So, it is necessary to know how to transform the CM wave function to an arbitrary reference frame. The transformation law of the bound state wave function from the rest frame to the moving one with four-momentum P was derived in the Bethe-Salpeter approach in Ref.[19] and in the quasipotential method in Ref.[20]. We use the last one and write the necessary transformation as follows:

$$\Psi_P^{\rho\omega}(\mathbf{p}) = D_1^{1/2,\ \rho\alpha}(R_{L_P}^W) D_2^{1/2,\ \omega\beta}(R_{L_P}^W) \Psi_0^{\alpha\beta}(\mathbf{p}), \tag{6}$$

$$\bar\Psi_P^{\lambda\sigma}(\mathbf{p}) = \bar\Psi_0^{\varepsilon\tau}(\mathbf{p}) D_1^{+\ 1/2,\ \varepsilon\lambda}(R_{L_P}^W) D_2^{+\ 1/2,\ \tau\sigma}(R_{L_P}^W),$$

where R^W is the Wigner rotation, L_P is the Lorentz boost from the meson rest frame to a moving one, and the rotation matrix $D^{1/2}(R)$ is defined by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} D_{1,2}^{1/2}(R_{L_P}^W) = S^{-1}(\mathbf{p}_{1,2}) S(\mathbf{P}) S(\mathbf{p}), \tag{7}$$

where the explicit form for the Lorentz transformation matrix of the four-spinor is

$$S(\mathbf{p}) = \sqrt{\frac{\epsilon(p)+m}{2m}} \left(1 + \frac{(\boldsymbol{\alpha}\mathbf{p})}{\epsilon(p)+m}\right). \tag{8}$$

For further transformation of the amplitude (2) the following relations are applied:

$$S_{\alpha\beta}(\Lambda) u_\beta^\lambda(p) = \sum_{\sigma=\pm 1/2} u_\alpha^\sigma(\Lambda p) D_{\sigma\lambda}^{1/2}(R_{\Lambda p}^W), \tag{9}$$

$$\bar u_\beta^\lambda(p) S_{\beta\alpha}^{-1}(\Lambda) = \sum_{\sigma=\pm 1/2} D_{\lambda\sigma}^{+\ 1/2}(R_{\Lambda p}^W) \bar u_\alpha^\sigma(\Lambda p).$$

Using also the transformation property of the Dirac bispinors to the rest frame

$$\bar u_1(\mathbf{p}) = \bar u_1(0) \frac{(\hat p_1' + m_1)}{\sqrt{2\epsilon_1(\mathbf{p})(\epsilon_1(\mathbf{p}) + m_1)}}, \quad p_1' = (\epsilon_1, \mathbf{p}),$$

$$v_2(-\mathbf{p}) = \frac{(\hat p_2' - m_2)}{\sqrt{2\epsilon_2(\mathbf{p})(\epsilon_2(\mathbf{p}) + m_2)}} v_2(0), \quad p_2' = (\epsilon_2, -\mathbf{p}), \tag{10}$$

we can introduce the projection operators $\hat\Pi^{\mathcal{P},\mathcal{V}}$ onto the states $(Q_1\bar Q_2)$ in the meson with total spin 0 and 1 as follows:

$$\hat\Pi^{\mathcal{P},\mathcal{V}} = [v_2(0)\bar u_1(0)]_{S=0,1} = \gamma_5(\hat\epsilon^*)\frac{1+\gamma^0}{2\sqrt{2}}. \tag{11}$$

As a result the doubly heavy meson production amplitude from e^+e^- annihilation takes the form:

$$\mathcal{M}_1(p_-, p_+, P, Q) = \frac{8\pi^2\alpha\alpha_s}{3s} \bar v(p_+)\gamma^\beta u(p_-) \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{\bar\Psi_0(\mathbf{p})}{\sqrt{2\epsilon_1(p)(\epsilon_1(p)+m_1)2\epsilon_2(p)(\epsilon_2(p)+m_2)}} \times \tag{12}$$

$$\times\int\frac{d\mathbf{q}}{(2\pi)^3}\frac{\bar\Psi_0(\mathbf{q})}{\sqrt{2\epsilon_1(q)(\epsilon_1(q)+m_1)2\epsilon_2(q)(\epsilon_2(q)+m_2)}}Sp\Big\{(\hat{\tilde{p}}_2-m_2)\hat{\tilde{\epsilon}}^*(1+\hat v_1)(\hat{\tilde{p}}_1+m_1)\times$$

$$\times\gamma_\mu\frac{(\hat r-\hat q_1+m_1)}{(r-q_1)^2-m_1^2+i\epsilon}\gamma_\beta(\hat{\tilde{q}}_1-m_1)\gamma_5(1+\hat v_2)(\hat{\tilde{q}}_2+m_2)\gamma_\nu\Big\}D_{\mu\nu}(k),$$

where the four-vectors $\tilde\epsilon$, $\tilde p_{1,2}$, $\tilde q_{1,2}$ are given by:

$$\tilde\epsilon=L_P(0,\boldsymbol{\epsilon})=\left(\boldsymbol{\epsilon}\mathbf{v},\boldsymbol{\epsilon}+\frac{(\boldsymbol{\epsilon}\mathbf{v})\mathbf{v}}{1+v^0}\right),\tag{13}$$

$$\hat{\tilde{p}}_{1,2}=S(L_P)\hat p'_{1,2}S^{-1}(L_P),\ \ S(L_P)(1\pm\gamma^0)S^{-1}(L_P)=\pm(\hat v_1\pm1),\ \ \hat v_1=\frac{\hat P}{M_{\mathcal V}},$$

$$\hat{\tilde{q}}_{1,2}=S(L_Q)\hat q'_{1,2}S^{-1}(L_Q),\ \ S(L_Q)(1\pm\gamma^0)S^{-1}(L_Q)=\pm(\hat v_2\pm1),\ \ \hat v_2=\frac{\hat Q}{M_{\mathcal P}}.$$

In order to make optimum use of the expression (12), we rearrange the bispinor contractions in the numerator of Eq.(12) and thus extracting in a more evident form the relative momenta p and q of heavy quarks:

$$\mathcal{M}_1(p_-,p_+,P,Q)=\frac{8\pi^2\alpha\alpha_s}{3s}\bar v(p_+)\gamma^\beta u(p_-)\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{\bar\Psi_0^{\mathcal V}(\mathbf{p})}{\sqrt{\frac{\epsilon_1(p)}{m_1}\frac{(\epsilon_1(p)+m_1)}{2m_1}\frac{\epsilon_2(p)}{m_2}\frac{(\epsilon_2(p)+m_2)}{2m_2}}}\times\tag{14}$$

$$\times\int\frac{d\mathbf{q}}{(2\pi)^3}\frac{\bar\Psi_0^{\mathcal P}(\mathbf{q})}{\sqrt{\frac{\epsilon_1(q)}{m_1}\frac{(\epsilon_1(q)+m_1)}{2m_1}\frac{\epsilon_2(q)}{m_2}\frac{(\epsilon_2(q)+m_2)}{2m_2}}}Sp\left\{\left[\frac{\hat v_1-1}{2}+\hat v_1\frac{\mathbf{p}^2}{2m_2(\epsilon_2(p)+m_2)}-\frac{\hat p}{2m_2}\right]\hat{\tilde{\epsilon}}^*(1+\hat v_1)\times\right.$$

$$\times\left[\frac{\hat v_1+1}{2}+\hat v_1\frac{\mathbf{p}^2}{2m_1(\epsilon_1(p)+m_1)}+\frac{\hat p}{2m_1}\right]\gamma_\mu\frac{(\hat r-\hat q_1+m_1)}{(r-q_1)^2-m_1^2+i\epsilon}\gamma_\beta D_{\mu\nu}(k)\times$$

$$\left.\times\left[\frac{\hat v_2-1}{2}+\hat v_2\frac{\mathbf{q}^2}{2m_1(\epsilon_1(q)+m_1)}+\frac{\hat q}{2m_1}\right]\gamma_5(1+\hat v_2)\left[\frac{\hat v_2+1}{2}+\hat v_2\frac{\mathbf{q}^2}{2m_2(\epsilon_2(q)+m_2)}-\frac{\hat q}{2m_2}\right]\gamma_\nu\right\}.$$

The production amplitude (14) contains relativistic corrections of two species. The first type of corrections are held in the quark interaction operator. They can be accounted for by means of the numerical solution of the Schrödinger equation with the relevant potential. The second type corrections are determined by several functions depending on the momenta of relative motion of quarks $\mathbf{p}$ and $\mathbf{q}$ including the gluon propagator $D_{\mu\nu}(k)$, relativistic bispinor factors and the heavy quark propagator. At least, there exist bound state corrections in Eq.(14) which are related to Eqs.(3) and (4).

Notice that there are several energy scales which can characterize the heavy quarkonium: the hard momentum scale m_Q (the mass of heavy quark), the soft momentum scale m_Qv_Q (v_Q is the heavy quark velocity in the bound state) and the ultrasoft momentum scale $m_Qv_Q^2$. We assume that the heavy quarkonium is a nonrelativistic system. This means that the following inequalities occur: $m_Q\gg m_Qv_Q\gg m_Qv_Q^2$, $m_Q\gg\Lambda_{QCD}$, which we can exploit in the further study of the production amplitude (14). The expansion of basic factors over relative momenta $\mathbf{p}$ and $\mathbf{q}$ up to terms of the second order has the following form:

$$\frac{1}{k^2}=\frac{1}{Z_1}\left[1-\frac{p^2+q^2}{Z_1}+\frac{2[\rho_2(rp)+\eta_2(rq)]}{Z_1}+\frac{4[\rho_2^2(rp)^2+\eta_2^2(rq)^2]}{Z_1^2}\right],\tag{15}$$

$$\frac{1}{(r-q_1)^2-m_1^2}=\frac{1}{Z_2}\left[1-\frac{q^2}{Z_2}+\frac{2(rq)}{Z_2}+\frac{4(rq)^2}{Z_2^2}\right],\quad Z_1=\eta_2\rho_2 r^2,$$
$$Z_2=r^2-2\rho_1(rQ)+\rho_1^2M_{\mathcal{P}}^2-m_1^2,\quad r=P+Q,$$

where we have chosen the center-of-mass frame with the condition $\mathbf{P}+\mathbf{Q}=0$; $r_0=\sqrt{s}$ is the total energy in the electron-positron annihilation. In the leading order, when one neglects relativistic and bound state corrections, one obtains $Z_1=(1-\kappa)^2 s$, $Z_2=(1-\kappa)s$, $\kappa=m_1/M_0=m_1/(m_1+m_2)$. So, the expansions (15) are well defined. Substituting (15) into Eq.(14) we have to combine the factors of the second degree over p and q. After averaging the obtained expression over the angle variables with the account of orthogonality conditions (3) and (4) [17], we can present the necessary correction in the integral form $\int d\mathbf{p}\mathbf{p}^2\bar\Psi_0^{\mathcal{V}}(\mathbf{p})$. Moreover, we performed the expansion of the amplitude (14) at $\mathbf{p}=\mathbf{q}=0$ over the binding energies $W_{\mathcal{P}}$ and $W_{\mathcal{V}}$ in the linear approximation in such a way that relativistic and bound state contributions can exist separately as follows:

$$\mathcal{M}_1(p_-,p_+,P,Q)=\frac{8\pi^2\alpha\alpha_s}{3}\bar v(p_+)\gamma^\beta u(p_-)\int\frac{d\mathbf{p}}{(2\pi)^3}\bar\Psi_0^{\mathcal{V}}(\mathbf{p})\int\frac{d\mathbf{q}}{(2\pi)^3}\bar\Psi_0^{\mathcal{P}}(\mathbf{q})\times \tag{16}$$
$$\times\epsilon_{\sigma\rho\lambda\beta}v_1^\sigma v_2^\rho\tilde\epsilon^{*\ \lambda}\frac{M_0}{s^3(1-\kappa)^3}\Biggl\{1+\frac{2W_{\mathcal{P}}}{M_0(1-\kappa)}(1-2\kappa+\kappa\frac{M_0^2}{s})+\frac{W_{\mathcal{V}}}{M_0(1-\kappa)}(2-3\kappa-2\kappa\frac{M_0^2}{s})+$$
$$\frac{\mathbf{p}^2}{72M_0^2\kappa^2(1-\kappa)^2}\left[-9+8\kappa\left(4+\frac{M_0}{\sqrt{s}}\right)-8\kappa^2(1-14\frac{M_0^2}{s}+10\frac{M_0^4}{s^2}\Bigr)\right]+\frac{\mathbf{q}^2}{72M_0^2\kappa^2(1-\kappa)^2}\times$$
$$\left[-9+16\kappa(1+\frac{M_0}{\sqrt{s}}-16\kappa^3\frac{M_0^2}{s}\left(3-\frac{4M_0}{\sqrt{s}}+\frac{14M_0^2}{s}\right)+32\kappa^2\frac{M_0}{\sqrt{s}}\left(-1+\frac{M_0}{\sqrt{s}}-\frac{2M_0^2}{s}+\frac{2M_0^3}{s^{3/2}}\right)\right]\Biggr\}.$$

Similar expressions occur for the other amplitudes contributing to the production process:

$$\mathcal{M}_2(p_-,p_+,P,Q)=\frac{8\pi^2\alpha\alpha_s}{3}\bar v(p_+)\gamma^\beta u(p_-)\int\frac{d\mathbf{p}}{(2\pi)^3}\bar\Psi_0^{\mathcal{V}}(\mathbf{p})\int\frac{d\mathbf{q}}{(2\pi)^3}\bar\Psi_0^{\mathcal{P}}(\mathbf{q})\times \tag{17}$$
$$\times\frac{M_0}{s^3(1-\kappa)^3}\Biggl\{1+\frac{2W_{\mathcal{P}}}{M_0}(1-\frac{M_0^2}{s})-\frac{W_{\mathcal{V}}}{M_0\kappa(1-\kappa)}(1+\kappa(-2-2\kappa+\kappa^2+2(1-\kappa)\frac{M_0^2}{s})+$$
$$\frac{\mathbf{p}^2}{72M_0^2\kappa^2(1-\kappa)^2}\Biggl[-9+40\kappa\left(1+\frac{M_0}{\sqrt{s}}\right)+4\kappa^2(1-18\frac{M_0}{\sqrt{s}}+24\frac{M_0^2}{s}-16\frac{M_0^3}{s^{3/2}}+16\frac{M_0^4}{s^2}\Bigr)-$$
$$-4\kappa^3\left(7-4\frac{M_0}{\sqrt{s}}+28\frac{M_0^2}{s}-16\frac{M_0^3}{s^{3/2}}+56\frac{M_0^4}{s^2}\right)\Biggr]+\frac{\mathbf{q}^2}{72M_0^2\kappa^2(1-\kappa)^2}\times$$
$$\times\left[-9-8\kappa(-8+\frac{M_0}{\sqrt{s}})-8\kappa^2\left(5-2\frac{M_0}{\sqrt{s}}-2\frac{M_0^2}{s}+4\frac{M_0^4}{s^2}\right)\right]\Biggr\}\epsilon_{\sigma\rho\lambda\beta}v_1^\sigma v_2^\rho\tilde\epsilon^{*\ \lambda},$$

$$\mathcal{M}_3(p_-,p_+,P,Q)=\frac{8\pi^2\alpha\alpha_s}{3}\bar v(p_+)\gamma^\beta u(p_-)\int\frac{d\mathbf{p}}{(2\pi)^3}\bar\Psi_0^{\mathcal{V}}(\mathbf{p})\int\frac{d\mathbf{q}}{(2\pi)^3}\bar\Psi_0^{\mathcal{P}}(\mathbf{q})\times \tag{18}$$
$$\times\frac{M_0}{s^3\kappa^3}\Biggl\{1+\frac{W_{\mathcal{P}}}{M_0\kappa}\left(-1-\frac{2M_0^2}{s}+2\kappa(2+\frac{M_0^2}{s})\right)+\frac{W_{\mathcal{V}}}{M_0\kappa}\left(-2+\frac{2M_0^2}{s}(1-\kappa)+2\kappa+\kappa^2\right)+$$
$$\frac{\mathbf{p}^2}{72M_0^2\kappa^2(1-\kappa)^2}\Biggl[7-16\frac{M_0}{\sqrt{s}}-16\frac{M_0^2}{s}-160\frac{M_0^4}{s^2}+4\kappa\left(5+10\frac{M_0}{\sqrt{s}}+36\frac{M_0^2}{s}-16\frac{M_0^3}{s^{3/2}}+136\frac{M_0^4}{s^2}\right)+$$

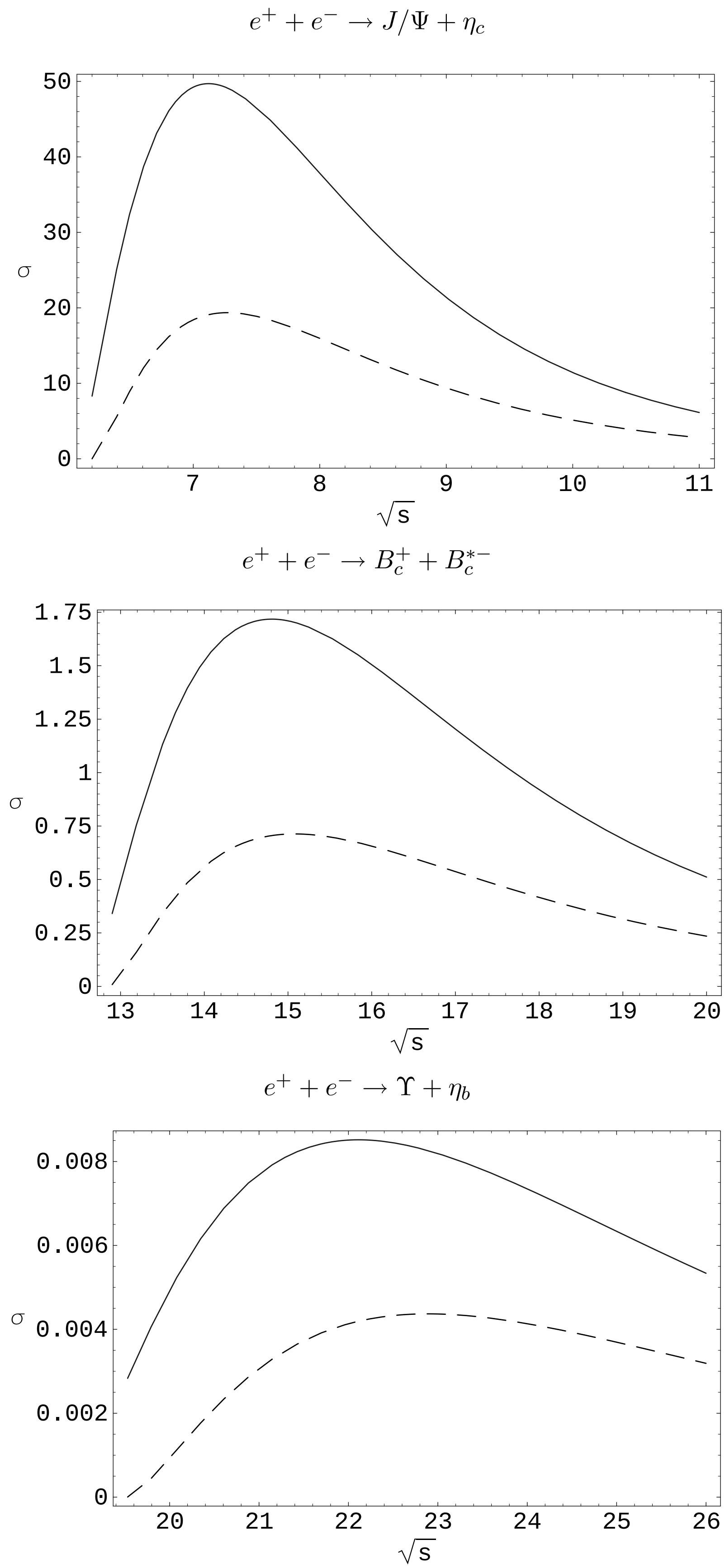

FIG. 2: The cross section in fb of e^+e^- annihilation into a pair of doubly heavy mesons with opposite charge parity as a function of the center-of-mass energy $\sqrt{s}$ (solid line). The dashed line shows the nonrelativistic result without bound state and relativistic corrections.

$$+8\kappa^2\left(4-3\frac{M_0}{\sqrt{s}}+30\frac{M_0^2}{s}-16\frac{M_0^3}{s^{3/2}}+76\frac{M_0^4}{s^2}\right)+4\kappa^3\left(-1-12\frac{M_0}{\sqrt{s}}+28\frac{M_0^2}{s}-16\frac{M_0^3}{s^{3/2}}+56\frac{M_0^4}{s^2}\right)\Bigg]+$$

$$+\frac{\mathbf{q}^2}{72M_0^2\kappa^2(1-\kappa)^2}\Bigg[15+8\frac{M_0}{\sqrt{s}}+16\frac{M_0^2}{s}-32\frac{M_0^4}{s^2}+8\kappa\left(2-3\frac{M_0}{\sqrt{s}}-4\frac{M_0^2}{s}+8\frac{M_0^4}{s^2}\right)-$$

$$-8\kappa^2\left(5-2\frac{M_0}{\sqrt{s}}-2\frac{M_0^2}{s}+4\frac{M_0^4}{s^2}\right)\Bigg]\Bigg\}\epsilon_{\sigma\rho\lambda\beta}v_1^\sigma v_2^\rho\tilde{\epsilon}^{*\ \lambda},$$

$$\mathcal{M}_4(p_-,p_+,P,Q)=\frac{8\pi^2\alpha\alpha_s}{3}\bar{v}(p_+)\gamma^\beta u(p_-)\int\frac{d\mathbf{p}}{(2\pi)^3}\bar{\Psi}_0^{\mathcal{V}}(\mathbf{p})\int\frac{d\mathbf{q}}{(2\pi)^3}\bar{\Psi}_0^{\mathcal{P}}(\mathbf{q})\times \tag{19}$$

$$\times\frac{M_0}{s^3\kappa^3}\Bigg\{1+\frac{W_{\mathcal{P}}}{M_0\kappa}\left(-2+\frac{2M_0^2}{s}+2\kappa(2-\frac{M_0^2}{s})\right)+\frac{W_{\mathcal{V}}}{M_0\kappa}\left(-1-\frac{2M_0^2}{s}+\kappa(3+\frac{2M_0^2}{s})\right)+$$

$$\frac{\mathbf{p}^2}{72M_0^2\kappa^2(1-\kappa)^2}\Bigg[15+8\frac{M_0}{\sqrt{s}}+16\frac{M_0^2}{s}-32\frac{M_0^4}{s^2}+8\kappa\left(-2-\frac{M_0}{\sqrt{s}}-4\frac{M_0^2}{s}+8\frac{M_0^4}{s^2}\right)-$$

$$-8\kappa^2\left(1-2\frac{M_0^2}{s}+4\frac{M_0^4}{s^2}\right)\Bigg]+\frac{\mathbf{q}^2}{72M^2\kappa^2(1-\kappa)^2}\Bigg[7-16\frac{M_0}{\sqrt{s}}-16\frac{M_0^2}{s}-160\frac{M_0^4}{s^2}+$$

$$+16\kappa\left(-1+3\frac{M_0}{\sqrt{s}}+5\frac{M_0^2}{s}-4\frac{M_0^3}{s^{3/2}}+34\frac{M_0^4}{s^2}\right)-16\kappa^2\frac{M_0}{\sqrt{s}}\left(2+7\frac{M_0}{\sqrt{s}}-8\frac{M_0^2}{s}+38\frac{M_0^3}{s^{3/2}}\right)+$$

$$+16\kappa^3\frac{M_0^2}{s}\left(3-4\frac{M_0}{\sqrt{s}}+14\frac{M_0^2}{s}\right)\Bigg]\Bigg\}\epsilon_{\sigma\rho\lambda\beta}v_1^\sigma v_2^\rho\tilde{\epsilon}^{*\ \lambda}.$$

It is important to emphasize that we carried out formal expansions of the integrands in the amplitudes $\mathcal{M}_i$ (16)-(19) over quantities $\mathbf{p^2}/m_{1,2}^2$, $\mathbf{q^2}/m_{1,2}^2$ assuming that the relative momenta of heavy quarks in the bound state are small as compared with their masses. This means that the formally divergent integrals $\int d\mathbf{p}\mathbf{p^2}\bar{\Psi}(\mathbf{p})$ and $\int d\mathbf{q}\mathbf{q^2}\bar{\Psi}(\mathbf{q})$ have to be defined by a suitable regularization procedure. Their numerical values will be directly determined by the properties of the wave functions of the heavy quark bound states.

III. CROSS SECTION OF PSEUDOSCALAR AND VECTOR DOUBLY HEAVY MESONS PRODUCTION

Using the relativistic amplitudes $\mathcal{M}_i$ obtained above, we can calculate the production cross section for doubly heavy mesons in e^+e^- annihilation as a function of the center-of-mass energy $\sqrt{s}$. For this purpose, the sum of amplitudes $\mathcal{M}_i$ (16)-(19) can be represented as follows:

$$\mathcal{M}(e^+e^-\to\mathcal{P}+\mathcal{V})=\frac{16}{27}\pi^2\alpha\frac{M_0\sqrt{4M_{\mathcal{P}}M_{\mathcal{V}}}}{s^6\kappa^5(1-\kappa)^5}\bar{\Psi}_0^{\mathcal{V}}(0)\bar{\Psi}_0^{\mathcal{P}}(0)\times \tag{20}$$

$$\times\bar{v}(p_+)\gamma^\beta u(p_-)\epsilon_{\sigma\rho\lambda\beta}v_1^\sigma v_2^\rho\tilde{\epsilon}^{*\ \lambda}\left[\kappa^3Q_1\alpha_{s2}T_1+(1-\kappa)^3Q_2\alpha_{s1}T_2\right],$$

where $\alpha_{s1}=\alpha_s(4m_1^2)$, $\alpha_{s2}=\alpha_s(4m_2^2)$, $Q_{1,2}$ are the electric charges of heavy quarks, $\Psi_0^{\mathcal{V},\mathcal{P}}(0)$ are the wave functions for the relative motion of heavy quarks in the vector and pseudoscalar meson at the origin in its rest frame,

$$T_1=72\kappa^2(1-\kappa)^2+\frac{\langle\mathbf{p}^2\rangle}{M_0^2}\Big[-9+36\kappa-2\kappa^2-14\kappa^3+\frac{M_0}{\sqrt{s}}4\kappa(6-9\kappa+2\kappa^2)+ \tag{21}$$

$$+\frac{M_0^2}{s}56\kappa^2(1-\kappa)+\frac{M_0^3}{s^{3/2}}32\kappa^2(\kappa-1)+\frac{M_0^4}{s^2}16\kappa^2(1-7\kappa)\Big]+\frac{\langle\mathbf{q}^2\rangle}{M_0^2}\Big[-9+40\kappa-20\kappa^2+$$

$$+\frac{M_0}{\sqrt{s}}4\kappa(1-2\kappa)+\frac{M_0^2}{s}24\kappa^2(1-\kappa)+\frac{M_0^3}{s^{3/2}}32\kappa^2(\kappa-1)+\frac{M_0^4}{s^2}16\kappa^2(1-7\kappa)\Big]+$$

$$+\frac{W_{\mathcal{V}}}{M_0}36\kappa\left[1-5\kappa^2+6\kappa^3-\kappa^4+\frac{M_0^2}{s}2\kappa(1-\kappa)^2\right]+\frac{W_{\mathcal{P}}}{M_0}72\kappa^2\left[2-5\kappa+3\kappa^2-\frac{M_0^2}{s}(1-\kappa)^2\right],$$

$$T_2=72\kappa^2(1-\kappa)^2+\frac{\langle\mathbf{p}^2\rangle}{M_0^2}\Big[11+2\kappa-20\kappa^2-2\kappa^3+\frac{M_0}{\sqrt{s}}4(-1+4\kappa+3\kappa^2-6\kappa^3)+ \tag{22}$$

$$+\frac{M_0^2}{s}56\kappa(1-\kappa)^2-\frac{M_0^3}{s^{3/2}}32\kappa(\kappa-1)^2+\frac{M_0^4}{s^2}16(-6+19\kappa-20\kappa^2+7\kappa^3)\Big]+\frac{\langle\mathbf{q}^2\rangle}{M_0^2}\Big[11-20\kappa^2+$$

$$+\frac{M_0}{\sqrt{s}}4(-1+3\kappa-2\kappa^2)+\frac{M_0^2}{s}24\kappa(1-\kappa)^2-\frac{M_0^3}{s^{3/2}}32\kappa(\kappa-1)^2+\frac{M_0^4}{s^2}16(-6+19\kappa-20\kappa^2+7\kappa^3)\Big]+$$

$$+\frac{W_{\mathcal{V}}}{M_0}36\kappa(-3+11\kappa-12\kappa^2+3\kappa^3+\kappa^4)+\frac{W_{\mathcal{P}}}{M_0}36\kappa(-3+14\kappa-19\kappa^2+8\kappa^3),$$

where $\langle\mathbf{p}^2\rangle$, $\langle\mathbf{q}^2\rangle$ are quantities determining the numerical values of relativistic effects connected with the internal motion of the heavy quarks in vector and pseudoscalar doubly heavy mesons. Note that they are not equal to the matrix elements $\int d\mathbf{p}\bar\Psi_0^{\mathcal{V}}(\mathbf{p})\mathbf{p}^2\Psi_0^{\mathcal{V}}(\mathbf{p})$ and $\int d\mathbf{q}\bar\Psi_0^{\mathcal{P}}(\mathbf{q})\mathbf{q}^2\Psi_0^{\mathcal{P}}(\mathbf{q})$ and are discussed at the end of Sec. III.

Performing standard algebraic calculations with the squared modulus of the amplitude $|\mathcal{M}|^2$ by the use of the system Form [21], we obtain the following differential cross section:

$$\frac{d\sigma}{d\cos\theta}=\frac{4\pi^3\alpha^2M_0^2|\Psi_0^{\mathcal{V}}(0)|^2|\Psi_0^{\mathcal{P}}(0)|^2}{729M_{\mathcal{V}}M_{\mathcal{P}}s^8\kappa^{10}(1-\kappa)^{10}}\left[\kappa^3Q_1\alpha_{s2}T_1+(1-\kappa)^3Q_2\alpha_{s1}T_2\right]^2\times \tag{23}$$

$$\times\left\{\frac{[s^2-(M_{\mathcal{V}}+M_{\mathcal{P}})^2][s^2-(M_{\mathcal{V}}-M_{\mathcal{P}})^2]}{s^4}\right\}^{3/2}\left(1+\cos^2\theta\right),$$

where θ is the angle between the momenta of the incoming lepton and the outgoing doubly heavy meson. The total cross section for the exclusive production of pseudoscalar and vector doubly heavy mesons in e^+e^- annihilation is then given by the following expression:

$$\sigma(s)=\frac{32\pi^3\alpha^2M_0^2|\Psi_0^{\mathcal{V}}(0)|^2|\Psi_0^{\mathcal{P}}(0)|^2}{2187M_{\mathcal{V}}M_{\mathcal{P}}s^8\kappa^{10}(1-\kappa)^{10}}\left[\kappa^3Q_1\alpha_{s2}T_1+(1-\kappa)^3Q_2\alpha_{s1}T_2\right]^2\times \tag{24}$$

$$\times\left\{\left[1-\frac{(M_{\mathcal{V}}+M_{\mathcal{P}})^2}{s^2}\right]\left[1-\frac{(M_{\mathcal{V}}-M_{\mathcal{P}})^2}{s^2}\right]\right\}^{3/2},$$

If we neglect relativistic and bound state effects in Eq.(24), that is $\langle\mathbf{p}^2\rangle=\langle\mathbf{q}^2\rangle=0$, $M_{\mathcal{P}}=M_{\mathcal{V}}=M_0$, then we obtain $\sigma_{NR}(s)$ coinciding with the analytical result of Ref.[22]. To estimate how the relativistic and bound state corrections can change the nonrelativistic cross section $\sigma_{NR}(s)$, we have used definite numerical values of a number of parameters entering in Eqs. (21)-(24). They can be determined on the basis of the relativistic quark model [23, 24, 25] as we shall discuss in the following.

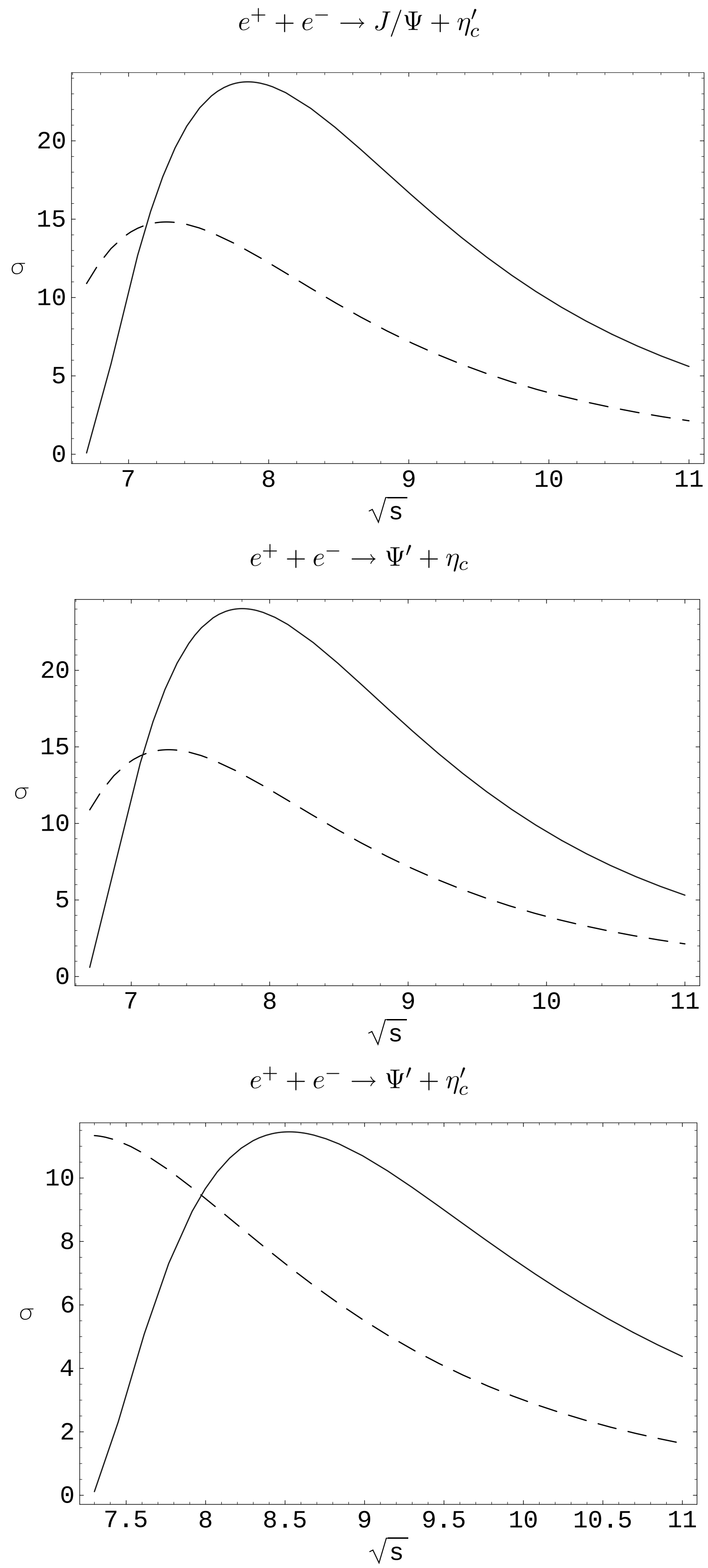

FIG. 3: The cross section in fb of e^+e^- annihilation into a pair of S-wave doubly charm heavy mesons with opposite charge parity as a function of the center-of-mass energy s (solid line). The dashed line shows the nonrelativistic result without bound state and relativistic corrections.

In the quasipotential approach the bound states of heavy quarks are described by the Schrödinger type equation [23]

$$\left(\frac{b^2(M)}{2\mu_R} - \frac{\mathbf{p}^2}{2\mu_R}\right)\Psi_0(\mathbf{p}) = \int \frac{d\mathbf{q}}{(2\pi)^3} V(\mathbf{p},\mathbf{q},M)\Psi_0(\mathbf{q}), \tag{25}$$

where the relativistic reduced mass is

$$\mu_R = \frac{E_1 E_2}{E_1 + E_2} = \frac{M^4 - (m_1^2 - m_2^2)^2}{4M^3}, \tag{26}$$

and the particle energies E_1, E_2 are given by

$$E_1 = \frac{M^2 - m_2^2 + m_1^2}{2M}, \quad E_2 = \frac{M^2 - m_1^2 + m_2^2}{2M}. \tag{27}$$

Here $M = E_1 + E_2$ is the bound state mass, $m_{1,2}$ are the masses of heavy quarks (Q_1 and Q_2) which form the meson, and $\mathbf{p}$ is their relative momentum. In the center of mass system the relative momentum squared on the energy surface $M = E_1 + E_2$ reads

$$b^2(M) = \frac{[M^2 - (m_1 + m_2)^2][M^2 - (m_1 - m_2)^2]}{4M^2}. \tag{28}$$

For small binding energies it is approximately equal to $2\mu_R W$. Notice that the kernel $V(\mathbf{p},\mathbf{q},M)$ in Eq.(25) is the quasipotential operator of the quark-antiquark interaction. The construction of the quark interaction operator is discussed permanently during the last decades [2, 3, 4, 26]. A new stage in solving this problem came when nonrelativistic field theories were introduced for the study of nonrelativistic bound states [27, 28, 29]. Within an effective field theory (NRQCD) the quark-antiquark potential was constructed in Refs.[29, 30, 31] by the perturbation theory improved renormalization group resummation of large logarithms. On the other hand, in the quasipotential quark model the kernel $V(\mathbf{p},\mathbf{q},M)$ is constructed phenomenologically with the help of the off-mass-shell scattering amplitude, projected onto the positive energy states. The heavy quark-antiquark potential with the account of retardation effects and the one-loop radiative corrections can then be presented in the form of a sum of spin-independent and spin-dependent parts. The explicit expression for it is given in Refs.[24, 25]. Taking into account the accuracy of the calculation of relativistic corrections to the cross section (24), we can use for the description of the bound system $(Q_1\bar{Q}_2)$ the following simplified interaction operator in the coordinate representation:

$$\tilde{V}(r) = -\frac{4}{3}\frac{\alpha_s(\mu^2)}{r} + Ar + B, \tag{29}$$

where the parameters of the linear potential $A = 0.18\ GeV^2$, $B = -0.3\ GeV$,

$$\alpha_s(\mu^2) = \frac{4\pi}{\beta_0 \ln(\mu^2/\Lambda^2)}, \quad \beta_0 = 11 - \frac{2}{3}n_f. \tag{30}$$

Here $n_f = 3$ is the number of flavours and $\mu = \frac{2m_1m_2}{(m_1+m_2)}$ is a renormalization scale and $\Lambda = 0.168$ GeV. All the parameters of the model like quark masses, parameters of the linear confining potential A and B, mixing coefficient ε and anomalous chromomagnetic quark moment κ entering in the quasipotential $V(\mathbf{p},\mathbf{q},M)$ were fixed from the analysis of heavy quarkonium masses [16, 23, 24, 25] and radiative decays [24]. Solving the Schrödinger-like quasipotential equation with the operator (29) we obtain an initial approximation for the bound state wave functions in the case of $(c\bar{c})$, $(\bar{b}c)$ and $(\bar{b}b)$ systems. For numerical estimations of relativistic and bound state effects in the production of heavy mesons in the e^+e^- annihilation we need the values of the wave functions at the origin, the bound state energy and the parameter of relativistic effects: $\int \mathbf{p}^2\bar{\Psi}_0^{\mathcal{P},\mathcal{V}}(\mathbf{p})d\mathbf{p}/(2\pi)^3$. This integral diverges at high momentum. Using the Schrödinger-like equation (25), we can express it through the value of the potential $V(r)$ at the origin. Different regularizations for it are discussed and already applied to a number of tasks concerning the production and decay of bound states [32, 33, 34, 35]. A recent discussion of the method for the calculation of an order-v^2 matrix element is presented in Ref.[36]. To estimate the numerical value of relativistic corrections in the production processes, we follow the prescription of dimensional regularization, where the scaleless momentum integral $\int V(\mathbf{p}-\mathbf{q})\Psi(\mathbf{q})\frac{d^d\mathbf{q}}{(2\pi)^d}\frac{d^d\mathbf{p}}{(2\pi)^d}$ related to the problem vanishes [32, 33, 35]. Then we can express the necessary quantity in the form:

$$\left\langle \mathbf{p}^2 \right\rangle_{DR} \equiv \frac{1}{\Psi(0)}\int \frac{d^d\mathbf{p}}{(2\pi)^d}\mathbf{p}^2\bar{\Psi}_0(\mathbf{p}) = 2\mu_R\tilde{W} + 2\mu_R|B|. \tag{31}$$

Let us note, that in the case of the Coulomb interaction in QED the dependence of the relativistic parameter (31) on the principal quantum number n is known in the analytical form [37]. For the quark bound states the dependence on the quantum number n is determined numerically. Solving the quasipotential equation (25) with the potential (29) [38], we obtain the energy spectrum $\tilde{W}$ of the heavy quark system and the numerical values of the parameter (31) for the bound states $(\bar{c}c)$, $(\bar{b}b)$ and $(\bar{b}c)$ which are presented in Table I. Heavy quark symmetry predicts that the wave functions of the vector and pseudoscalar states are different due to corrections of order v_Q^2. The analogous statement is valid for the parameter $\langle \mathbf{p}^2\rangle$. Neglecting in this study the corrections of order $O(v_Q^4)$ in the production rates we write in Table I equal values of $\langle \mathbf{p}^2\rangle$ for V- and P-mesons. The theoretical uncertainty of the obtained value of the parameter $\langle \mathbf{p}^2\rangle$ in Table I is determined by perturbative and nonperturbative corrections to the quasipotential [23, 24] and is not exceeding 30%. We presented in Table I the estimation of the pseudoscalar and vector wave functions at the origin including effects of order $O(v_Q^2)$. For this aim first of all, we obtained the central equal values of $\Psi_{\mathcal{V}}(0)$ and $\Psi_{\mathcal{P}}(0)$ solving the Schrödinger-like equation (25). Then we took into account the spin-dependent corrections of order $O(v_Q^2)$ to the wave functions at the origin which were calculated in the framework of nonrelativistic QCD (see Refs.[30, 39, 40, 41]).

IV. SUMMARY AND DISCUSSION

We have investigated the relativistic and bound state effects in the production of doubly heavy mesons on the basis of the perturbative QCD and the relativistic quasipotential quark model. Using the factorization hypothesis we keep systematically all relativistic corrections of the second order in the relative velocity of heavy quarks and bound state effects in the production of the S-wave pseudoscalar and vector mesons from e^+e^- annihilation.

Let us summarize the basic peculiarities of the calculation performed above.

1. We obtain the cross sections for the production of a pair of S-wave doubly heavy mesons with opposite charge parity containing b and c quarks from e^+e^- annihilation.

TABLE I: Basic parameters of the relativistic quark model

State $n^{2S+1}L_J$	Particle	Mass, GeV [24, 42]	α_s	Bound state energy, GeV	$\Psi(0)$, $GeV^{3/2}$	$\langle \mathbf{p}^2 \rangle_{DR}$, GeV^2
1^1S_0	η_c	2.980	0.314	-0.120	0.28	0.6
1^3S_1	J/Ψ	3.097	0.314	-0.003	0.26	0.6
2^1S_0	η_c'	3.594	0.314	0.494	0.25	1.4
2^3S_1	Ψ'	3.686	0.314	0.586	0.23	1.4
1^1S_0	B_c^+	6.270	0.265	-0.160	0.36	0.7
1^3S_1	B_c^{*-}	6.332	0.265	-0.098	0.34	0.7
2^1S_0	$B_c'^+$	6.835	0.265	0.405	0.34	1.8
2^3S_1	$B_c'^{*-}$	6.881	0.265	0.451	0.32	1.8
1^1S_0	η_b	9.400	0.207	-0.360	0.55	0.9
1^3S_1	Υ	9.460	0.207	-0.300	0.53	0.9
2^1S_0	η_b'	9.993	0.207	0.233	0.45	2.9
2^3S_1	Υ'	10.023	0.207	0.263	0.43	2.9

TABLE II: Comparison of theoretical predictions with experimental data.

State H_1H_2	$\sigma_{BaBar}\times$ $Br_{H_2\to charged\geq 2}$ (fb) [7]	$\sigma_{Belle}\times$ $Br_{H_2\to charged\geq 2}$ (fb) [6]	σ (fb) [14]	σ_{NRQCD} (fb) [8]	σ (fb) [10]	σ (fb) [8]	Our result (fb)
$\Psi(1S)\eta_c(1S)$	$17.6 \pm 2.8^{+1.5}_{-2.1}$	$25.6 \pm 2.8 \pm 3.4$	26.7	3.78	5.5	7.4	7.8
$\Psi(2S)\eta_c(1S)$		$16.3 \pm 4.6 \pm 3.9$	16.3	1.57	3.7	6.1	6.7
$\Psi(1S)\eta_c(2S)$	$16.4 \pm 3.7^{+2.4}_{-3.0}$	$16.5 \pm 3. \pm 2.4$	26.6	1.57	3.7	7.6	7.0
$\Psi(2S)\eta_c(2S)$		$16.0 \pm 5.1 \pm 3.8$	14.5	0.65	2.5	5.3	5.4

2. All possible sources of relativistic corrections including the transformation factors for the two quark bound state wave function have been taken into account.

3. We have investigated the role of relativistic and bound state effects in the total production cross sections using predictions of the relativistic quark model for a number of parameters entering in the obtained analytical expressions.

The results of our calculation of the cross section (24) presented in Figs.2-3 evidently show that only the relativistic analysis of the production processes can give reliable theoretical predictions for the comparison with the experimental data. It follows from Fig.3 that with the growth of the quantum number n the nonrelativistic approximation doesn't work near the production threshold because the omitted terms in this case have the same order of the magnitude as the basic terms.

As we have already mentioned, the experimental results for the production of $J/\Psi + \eta_c$ mesons measured at the Belle and BaBar experiments differ from theoretical calculations in the framework of NRQCD. The experimental data on the production cross sections of a pair of S-wave charm mesons are presented in Table II. The numerical value for the cross section of $J/\Psi + \eta_c$ production at $\sqrt{s} = 10.6$ GeV, obtained on the basis of Eq.(24) amounts to the value 7.8 fb without the inclusion of QED effects. In this case relativistic and bound state corrections increase our nonrelativistic result by a factor 2.2 (cf. dashed lines in Figs.2,3). Accounting slightly different values of several parameters used in our model in the comparison with Ref.[8] (the mass of c quark, the binding energies $W_{\mathcal{P},\mathcal{V}}$), we find agreement of our calculations with the results of Ref.[8] for the production of the charmonium states, if relativistic corrections are taken into account (see the seventh column of Table II). Taking in mind also the calculation [15], which includes additional perturbative corrections of order α_s, we observe the convergence between the experimental data and theoretical results obtained on the basis of approaches combining nonrelativistic QCD and the relativistic quark model. Our results should be useful for the future comparison with the $(b\bar{b})$ and $(c\bar{b})$ meson production measurements. Similarily, we can examine the production rates for doubly heavy mesons including the P- and D-wave states. The experimental data for the production of the mesons $\chi_{c0} + J/\Psi$ are obtained already in Refs.[6, 7]. The work in this direction is in progress.

There is an essential difference in the numerical results obtained in this work and in Refs.[12, 13, 14] on the basis of the light-cone formalism. Our results lead to the increase of the cross section for the production $J/\Psi + \eta_c$ only by a factor $2 \div 2.5$ in the range of center-of-mass energies $\sqrt{s} = 6 \div 12$ GeV but not to an order of magnitude at $\sqrt{s} = 10.6$ Gev as in Refs.[12, 13, 14]. While the cross section (24) depends on the choice of the c-quark mass, strong coupling constant α_s, the meson wave functions at the origin, binding energies $W_{\mathcal{P},\mathcal{V}}$ and the relativistic parameter (31), a possible growth of the theoretical value (24) doesn't solve the problem. So, there exist at least two questions which could be discussed regarding the performed calculation.

The first question refers to the comparison of our calculation with the light-cone approach to the same problem in Refs.[12, 13, 14]. Since the main part of the investigation was connected with the relative motion of heavy quarks in the meson, it is necessary to find the wave function $\psi(\mathbf{p}_\perp, x)$ which describes bound states in such an approach and satisfies the quasipotential wave equation in the light-front formalism [43, 44, 45] (see Appendix A). This equation gives the wave function of a bound state in an arbitrary Lorentz reference frame. The transformation property for the wave function from the arbitrary frame to the frame, in which the total transverse momentum of the two-particle bound state is equal to zero is determined by Eq.(A3). But contrary to our approach to the production of doubly heavy mesons, accounting the relative motion of the heavy quarks, the quark transverse momentum inside the heavy quarkonium was neglected in Refs.[12, 13, 14]. So, the basic difference between our method and the light-cone formalism consists in the fact that in Ref.[12, 13, 14] the effects of the wave function $\psi(x, \mathbf{p}_\perp)$ transformation from the frame $\mathbf{P}_\perp = 0$ to the moving one with the transverse momentum $\mathbf{P}_\perp \neq 0$ were not taken into account. They lead to terms proportional to the meson transverse momentum squared $\mathbf{P}_\perp^2$ in the production amplitude and might correct the total result.

Finally, another important question regards the accuracy of the calculation performed in this paper and the numerical estimation of the next term in the expansion of the production amplitude over the relative velocity of the fourth order. Recently, corrections of order $O(v^4)$ were studied in the $S-$ and P-wave quarkonium decays in Refs.[46, 47]. There it was demonstrated that the v expansion converges well for the decays of heavy quarkonium. Note that a numerical estimation of the parameter $\langle \mathbf{p}^4 \rangle$ can be obtained as in Eq.(31) by using the dimensional regularization technique [35]: $\left\langle \mathbf{p}^4/m_Q^4 \right\rangle_{DR} = \frac{(\tilde{W}+|B|)^2}{m_Q^2}$. When we consider the production of $1S$-wave charmonium states with the binding energy $W \ll m_Q$, then the numerical values of the parameters $\langle \mathbf{p}^2 \rangle_{DR}$, $\langle \mathbf{p}^4 \rangle_{DR}$ are small. In the production of excited S-wave states the binding energy increases what leads to the growth of $\langle \mathbf{p}^2 \rangle_{DR}$, $\langle \mathbf{p}^4 \rangle_{DR}$. So, it follows from the results of Table I that $\langle \mathbf{v}^2 \rangle_{DR} \approx 0.6$ for the $2S$-charmonium states and $\langle \mathbf{v}^2 \rangle_{DR} \approx 0.7$ for the $2S$ $(\bar{b}c)$ meson. Obviously, the convergence of the expansion becomes worse in this case. The calculation of the coefficient in the term $\langle \mathbf{v}^4 \rangle$ would deserve additional investigation. On the whole, we estimate the theoretical uncertainty of the obtained production rates due to inaccuracies in the determination of the meson wave functions at the origin and the omitted terms in the expansion of the relativistic amplitudes as the 40% for the $1S$-states and the 70% for the $2S$-states.

Acknowledgments

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APPENDIX A: QUASIPOTENTIAL LIGHT-CONE DISTRIBUTION AMPLITUDE

In this Appendix we consider the construction of the heavy quark light-cone distribution amplitude on the basis of the quasipotential equation derived in light-cone variables.

The quasipotential wave equation for the two-body bound state in the light-front formalism takes the form [43, 44, 45]:

$$\left[P^2 - \frac{\left(\mathbf{p}_\perp + \left(\frac{1}{2} - x\right)\mathbf{P}_\perp\right)^2 + m_1^2}{x} - \frac{\left(\mathbf{p}_\perp + \left(\frac{1}{2} - x\right)\mathbf{P}_\perp\right)^2 + m_2^2}{1-x}\right]\psi_P(\mathbf{p}_\perp, x) = \tag{A1}$$

$$= \int_0^1 \frac{dy}{y(1-y)} \int \frac{d\mathbf{q}_\perp}{2(2\pi)^3} V(P, x, \mathbf{p}_\perp; y, \mathbf{q}_\perp)\psi_P(\mathbf{q}_\perp, y),$$

where p $(\mathbf{p}_\perp)$ and P $(\mathbf{P}_\perp)$ are total (transverse) momenta of relative motion of quarks and the meson. The variable x is introduced in the following way:

$$x = \frac{1}{2} + \frac{p_+}{P_+}, \quad p_\pm = p_0 \pm p_3, \quad P_\pm = P_0 \pm P_3. \tag{A2}$$

This equation allows to find the two-particle wave function in an arbitrary Lorentz reference frame. The transformation of the wave function from the arbitrary frame to the frame, in which the total transverse momentum $\mathbf{P}_\perp = 0$ is the following:

$$\psi_P(x, \mathbf{p}_\perp) = \psi_{\mathbf{P}_\perp = 0}(x, \mathbf{p}_\perp + (1/2 - x)\mathbf{P}_\perp). \tag{A3}$$

The case of spin particles is discussed in Refs.[45, 48]. In the equal mass case ($m_1 = m_2 = m$) in the frame where $\mathbf{P}_\perp = 0$ we get:

$$\left[M^2 - \frac{\mathbf{p}_\perp{}^2 + m^2}{x(1-x)}\right]\psi(\mathbf{p}_\perp, x) = \int_0^1 \frac{dy}{y(1-y)}\int \frac{d\mathbf{q}_\perp}{2(2\pi)^3} V(x, \mathbf{p}_\perp; y, \mathbf{q}_\perp)\psi(\mathbf{q}_\perp, y). \tag{A4}$$

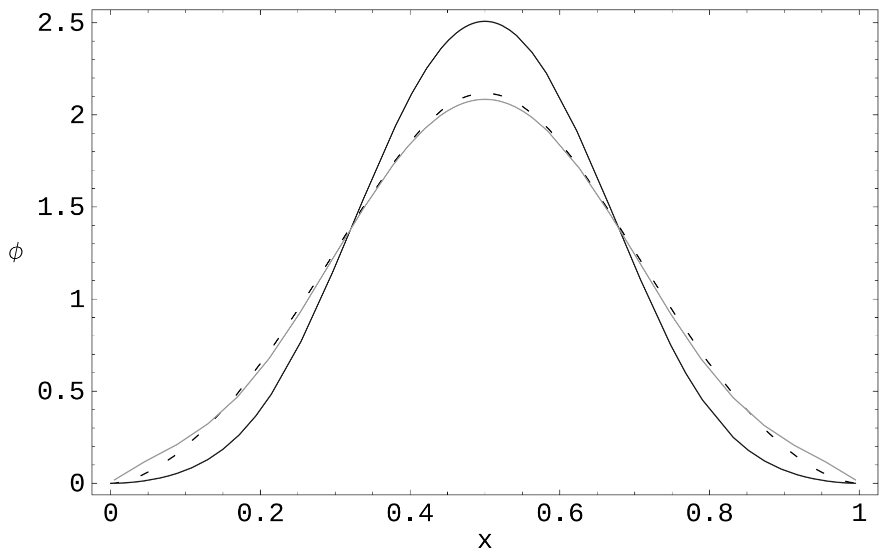

FIG. 4: The light-front distribution amplitude over the momentum fraction x (solid line). The dashed line shows the phenomenological function from Ref.[13]. The thin solid line shows the result obtained in our model by means of the relations (44), (49) of Ref.[49] with the c-quark mass $m_c = 1.55$ Gev.

In order to discuss the realistic wave function $\psi(\mathbf{p}_\perp, x)$ there is the need to transform Eq.(A4) to the ordinary form (25), because we know the potential (29) for it which describes the energy spectrum with sufficiently high accuracy. Let us introduce the relative momentum $\mathbf{p} = (\mathbf{p}_\perp, p_3) = (p\sin\theta\cos\phi, p\sin\theta\cos\phi, p\cos\theta)$:

$$p_3 = \frac{x - \frac{1}{2}}{\sqrt{x(1-x)}}\sqrt{\mathbf{p}_\perp{}^2 + m^2}, \quad x = \frac{1}{2}\left(1 + \frac{p_3}{\epsilon(p)}\right) = \frac{1}{2}\left(1 + \frac{p\cos\theta}{\sqrt{p^2+m^2}}\right), \tag{A5}$$

so that

$$dx d^2p_\perp = \frac{2x(1-x)}{\sqrt{p^2+m^2}} d\mathbf{p}. \tag{A6}$$

As a result, we obtain from Eq.(A4) the ordinary quasipotential equation (25), which contains the wave function $\Psi_0(\mathbf{p})$ connected with the initial one by the following relation:

$$\Psi_0(\mathbf{p}) = \left[\frac{4x(1-x)}{\mathbf{p}_\perp{}^2 + m^2}\right]^{1/4} \psi(\mathbf{p}_\perp, x). \tag{A7}$$

The ordinary normalization condition for the wave function $\Psi_0(\mathbf{p})$ can be rewritten for the function $\psi(\mathbf{p}_\perp, x)$ as follows:

$$\int_0^1 \frac{dx}{x(1-x)}\int \frac{d\mathbf{p}_\perp}{2(2\pi)^3}|\psi(\mathbf{p}_\perp, x)|^2 = 1. \tag{A8}$$

Then we determine the light-cone distribution amplitude by means of the quasipotential light-front function $\psi(\mathbf{p}_\perp, x)$:

$$\phi(x) = \mathcal{N} \int \psi(\mathbf{p}_\perp, x) d^2\mathbf{p}_\perp = \mathcal{N} \cdot 8\pi \cdot x(1-x) \int_{m\sqrt{\frac{(x-1/2)^2}{x(1-x)}}}^{\infty} (p^2+m^2)^{1/4} p \Psi_0(p) dp. \quad \text{(A9)}$$

The integral function in Eq.(A9) differs by the power of the relativistic energy $\epsilon(p)$ ($\epsilon^{1/4}(p) \rightarrow \epsilon^{1/2}(p)$) and the function $x(1-x)$ from the expression in Ref.[49]. The normalization factor $\mathcal{N}$ is determined by the condition: $\int_0^1 \phi(x) dx = 1$. The plot of the function $\phi(x)$ and the comparison with the corresponding amplitudes in Refs.[13, 49] are presented in Fig.4. All these functions, obtained in the frame where the meson transverse momentum $\mathbf{P}_\perp = 0$, have the similar shape. But our function (solid line in Fig.4) has a larger maximum and falls quicker to the end points $x = 0, 1$. To derive the light-cone distribution in the arbitrary reference frame we can use Eqs.(A3), (A9). The shift of the transverse momentum $\mathbf{p}_\perp \rightarrow \mathbf{p}_\perp - \left(\frac{1}{2} - x\right) \mathbf{P}_\perp$ shows that the quark light-cone distribution doesn't change the shape in the arbitrary frame with $\mathbf{P}_\perp \neq 0$. So, the light-cone amplitude transformation from dashed line of Ref.[13] to our solid line in Fig.4 decreases the production rates of the charmonium states in the light-front approach.

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